A characterization of Einstein manifolds

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Abstract

In this work we wish characterize the Einstein manifolds (M, g), however without the necessity of hypothesis of compactness over M and unitary volume of g, which are well known in many works. Our result says that if all eingenvalues λ of r_g, with respect to g, satisfy $\lambda \geq \frac{1}{n}s_g$, then (M, g) is an Einstein manifold, where r_g and s_g denote the Ricci and scalar curvatures, respectively.

Key words: Einstein manifolds; stochastic analysis on manifolds.

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1 Introduction

Let (M, g) be a Riemannian manifold. Let us denote the Ricci curvature by r_g and the scalar curvature by s_g. M is called an Einstein manifold if, for every vector fields X, Y on M, there exists a real constant such that

$$r_g(X, Y) = \lambda g(X, Y).$$

We call the metric g of Einstein metric. We recall that scalar curvature s_g is defined by $\operatorname{tr} r_g$. A simple account gives that, for Einstein metrics, s_g is the constant λn, where n is the dimension of M. For a fuller treatment we refer the reader to [1].

To characterize Einstein metrics we assume the fact that their scalar curvature is constant. However, we do not ask the compactness property over Riemannian manifolds.

Our main result is proved using stochastic analysis on manifolds. The basic idea is, from hypothesis of scalar curvature constat, show that the integral of the Ricci tensor $r_g - 1/n s_g g$ along any g-Brownian motion B_t in M is null, namely, $\int (r_g - 1/n s_g g)(dB, dB) = 0$. From this we found conditions, see Proposition 3.1, to conclude that $r_g - 1/n s_g g$ is null, that is, g is an Einstein metric. In summary, we state our Theorem.

Theorem : *Let (M, g) be a Riemannian manifold, r_g its Ricci curvature and s_g its scalar curvature. If all eigenvalues λ of r_g, with respect to g, satisfy $\lambda \geq \frac{1}{n}s_g$, where n is the dimension of M, then (M, g) is an Einstein manifold.*

This article is present in the following way: Section 2 contains a brief summary of Analysis Stochastic on Manifolds. In Section 3 our main results are stated and proved.

2 Stochastic tools

In the following we always consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$. We begin for introduce the three most important process for stochastic analysis in manifolds. See for instance [2] for a complete study about these process. From now on the term smooth means of class C^∞.

Definition 2.1 *Let M be a smooth manifold. A continuous M-valued process X_t is called semimartingale if, for each smooth f on M, the real-valued process $f \circ X_t$ is a semimartingale.*

Let X_t be a semimartingale on M and b be a bilinear form on M. Let $(U, x_1, \ldots, x_n)$ be a local coordinate system on M. In this coordinate b is written as $b_{ij} dx^i \otimes dx^j$, where b_{ij} are smooth function on U. The integral of b along X_t is defined, locally, by

$$\int b(dX, dX) = \int b_{ij}(X_t) d[X^i, X^j]_t, \tag{1}$$

where $X_t^i = x^i \circ X_t$, $i = 1, \ldots n$.

Using this definition has sense the following definition of martingales in smooth manifolds.

Definition 2.2 *Let M be a smooth manifold with a connection ∇. A semimartingale X_t in M is called a martingale if, for every smooth f,*

$$f \circ X_t - f \circ X_0 - \int \mathrm{Hess} f(dX, dX)$$

is a real local martingale. Here, Hess denotes the Hessian operator associated to connection ∇.

In the sequel, we define Brownian motion in a smooth manifold.

Definition 2.3 *Let (M, g) be a Riemannian manifold. Given $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, a M-valued process B_t is called a g-Brownian motion in (M, g) if B_t is continuous and adapted and, for every smooth f,*

$$f \circ B_t - f \circ B_0 - \frac{1}{2} \int \Delta_g f \circ B_t dt$$

is a real local martingale, where Δ_g is the Laplace-Beltrami operator associated to metric g.

Given a point x in (M, g), there always exists a g-Brownian motion B_t in M, starting at x, defined on $[0, \zeta[$ for some complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ and some stopping time $\zeta > 0$.

The following Lemma, which demonstration is found in [2, Lemma 5.20], is fundamental in the proof of our Theorem.

Lemma 2.1 *If B_t is a g-Brownian motion, then , for every bilinear form b,*

$$\int b(dB, dB) = \int \operatorname{tr} b(B_t) dt.$$

3 Einstein manifolds

We begin recalling the definition of Ricci tensor field for a metric and Einstein Manifold.

Definition 3.1 *The Ricci curvature r_g of a Riemannian manifold (M, g) is the 2-tensor*

$$r_g(X, Y) = \operatorname{tr}(Z \to R(X, Z)Y),$$

where tr *denotes the trace of the linear map $Z \to R(X, Z)Y$.*

Definition 3.2 *A Riemannian manifold (M, g) is Einstein if there exists a real constant λ such that*

$$r_g(X, Y) = \lambda g(X, Y).$$

We call the metric g of Einstein metric.

Before we prove our Theorem we show a general Proposition in Analysis Stochastic on Manifolds. This result is important because in the proof of Theorem we have an step that is a particular case of this Proposition. In fact, this is the final step to conclude the proof.

Proposition 3.1 *Let (M, g) be a Riemannian manifold. Let b a symmetric bilinear form in M such that their eigenvalues $\lambda_i \geq 0$, $i = 1, \ldots n$. If $\int b(dB, dB) = 0$ for some g-Brownian motion B_t, then $b = 0$.*

Proof: The proof is for a contrapositive argument. Let b be a symmetric bilinear form on M such that $b \geq 0$. Suppose that $b > 0$, that is, for $X, Y \in TM$, $b(X, Y) > 0$. From Lemma 2.1, for each g-Brownian motion B_t, we obtain that

$$\int b(dB, dB) = \int \operatorname{tr} b(B_t) dt = \sum_{i=1}^{n} \int \lambda_i(B_t) dt,$$

We see that for each $\omega \in \Omega$, $\lambda_i(B_t(\omega))$ is continuous, for each $i = 1, \ldots, n$. Since $\lambda_i(B_t) > 0$ and dt is the Lebesgue measure, it follows that $\int \lambda_i(B_t) dt > 0$. Thus we get $\int b(dB, dB) > 0$. $\qquad \square$

Now we prove our Theorem.

Theorem 3.2 *Let (M, g) be a Riemannian manifold, r_g its Ricci curvature and s_g its scalar curvature. If all eigenvalues λ of r_g, with respect to g, satisfy $\lambda \geq \frac{1}{n} s_g$, where n is the dimension of M, then (M, g) is an Einstein manifold.*

Proof: Let (M, g) be a Riemannian manifold. Suppose that s_g is constant, that is, there exists $c \in \mathbb{R}$ such that $s_g(x) = c$ for all $x \in M$. If n is dimension of M, then

$$s_g(x) = \frac{\lambda}{n} n.$$

As $n = \operatorname{tr} g$ we have

$$s_g(x) - \frac{c}{n} \operatorname{tr} g(x) = 0.$$

Applying this equality about an arbitrary g-Brownian motion B_t in M we obtain

$$s_g(B_t) - \frac{c}{n} \operatorname{tr} g(B_t) = 0.$$

We now integrate in t each trajectory of B_t, that is,

$$\int \operatorname{tr}(r_g - \frac{c}{n} g)(B_t) dt = \int \operatorname{tr} r_g(B_t) - \frac{c}{n} \operatorname{tr} g(B_t) dt = 0.$$

From Lemma 2.1 we conclude that

$$\int (r_g - \frac{c}{n} g)(dB, dB) = 0. \tag{2}$$

We now observe that eigenvalues of symmetric bilinear form $r_g - \frac{c}{n} g$ are $\lambda_i - \frac{c}{n}$, where λ_i, $i = 1, \ldots, n$, are eigenvalues of r_g with respect to g. By hypothesis, $\lambda_i - \frac{c}{n} \geq 0$. Using this fact in Proposition 3.1 we conclude that $r_g = \frac{c}{n} g$. Therefore, g is an Einstein metric. $\square$

As a simple result from Theorem above we state an one about Ricci-Flat manifolds.

Corollary 3.3 *Under hypothesis of Theorem 3.2, if $s_g = 0$ then (M, g) is Ricci-Flat manifold.*

References

[1] Besse, Arthur L., *Einstein manifolds*. Reprint of the 1987 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008. xii+516 pp.

[2] Emery, M., *Stochastic Calculus in Manifolds*, Springer, Berlin 1989.